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FAST UNITARY TRANSFORMS-
BENEFITS AND RESTRICTIONS

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The basic facts about Walsh functions are presented, a number of applications are indicated, and their properties in orthogonal expansions are discussed. Fast discrete Walsh transforms is one and higher dimensional cases are described, and the attendant advantages and disadvantages in their use in Fourier expansions are pointed out.

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A number of methods exist which generalize the Fourier transform either by considering the class of Fast Unitary Transforms (which include the FFT) or by considering group characters. The practical interest here is in the computational efficiency inherent in these more general transforms.

For an arbitrary sequence of functions, the Gram-Schmidt process generates a sequence of orthogonal functions. Any continuous function can be expressed via this orthogonal set with minimum $L_2$ error; for certain classes of functions the convergence is uniform. Most inportant among these are the Haar functions, which assume 2 values, and the Walsh functions with values $\pm 1$. Both Both have discrete analogies and Fast Discrete Haar/Walsh transforms are computable, FHT and FWT. Because of the simplicity of the basic functions much greater computational savings can be obtained than from FFT (up to 30 times faster than FFT).

Advantageous use can be made of FHT/FWT in certain applications; e.g. in data transmission/reconstruction in which one represents a given signal in some sense and reconstruct it from the minimal representation. A number of serious difficulties arise with these transforms due to the fact that the relation with the circle has been lost. For example: 1) no natural interpretation in terms of frequency exists (the "sequency" viewpoint of Harmuth for Walsh functions lacks physical meaning); 2) due to absence of the circle relationship the important convolution theorem is not available (forced analogies to a convolution theorem via dyadic convolutions have been made but their interpretations are not clear).

The striking advantages of FWT and FHT over the usual FFT in computational effort should motivate further investigation in this area.

The historical situation regarding orthogonal functions at the beginning of the twentieth century was one of well known and useful kinds of such functions: the trigonometric functions which occur in Fourier series; orthogonal polynomials such as those of Legendre, Hermite, and Laguerre;

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Bessel"s functions, the Sturm-Liouville series, and other special functions. But, there was no general theory embracing all such systems of functions.

The Hungarian mathematician, Alfred Haar, was concerned with convergence properties of series of orthogonal functions, and also constructing a new set (now called the Haar system) of such functions. He defined a set of orthogonal functions each taking essentially only two values such that the formal expansion of an arbitrary continuous function in those functions converges uniformly to the given function, a property not possessed by orthogonal sets known up to that time.

In 1923, J.L. Walsh published a set of orthogonal functions which are complete on the interval [0.1]; they take only the values $\pm 1$, and are similar in oscillation and many other properties to the trigonometric functions. They have turned out to have important practical applications in calculation.

The limits of the usefulness of these functions both in theoretical work and in engineering applications still seem to be undetermined.

Traditionally, the theory of communication has been based on the complete, orthogonal system of sine and cosine functions. The concept of frequency is defined as the parameter f in $\sin 2\pi$ ft. and $\cos 2\pi$ ft. The question arises whether there are other systems of functions on which theories of similar scope can be based, and that lead to equipment of practical interest.

The parameter in $\sqrt{2}\sin 2i\pi\theta$ and $\sqrt{2}\cos 2i\pi\theta$ gives the number of oscillations in the interval $-1/2 \leq \theta < 1/2$ (that is, the normalized frequency i=fT). One may interpret i as "one half the number of zero crossings per unit time" rather than as "oscillations per unit time". (The zero crossing at the left side, $\theta = -1/2$, but not the one at the right side, $\theta = +1/2$, of the time interval is counted for sine functions).

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The parameter i also equals one half the number of zero crossings in the interval $-1/2 \leq \theta < 1/2$ for Walsh functions. In contrast to sine-cosine functions, the sign changes are not equidistant. If i is not an integer, then it equals "one half the average number of zero crossings per unit time". The term "normalized sequency" has been introduced for L and $\Phi = i/T$ is called the normalized sequency. Sequency in zps = 1/2 (average number of zero crossings per second).

The general form of a sine function $V \sin (2\pi ft + \alpha)$ contains the parameters amplitude V, frequency f, and phase angle $\alpha$. The general form of a Walsh function $V \text{ sal } (\Phi T, t/T + t_o/T)$ contains the parameters amplitude, V, sequency, $\Phi$, the delay, $t_o$, and time base, T. The normalized delay, $t_o/T$, corresponds to the phase angle. The time base, T, is an additional parameter and it causes a major part of the differences in the applications of sine-cosine and Walsh functions.

So far, Walsh functions are the only known functions with desirable features comparable to sine-cosine functions for use in communications. Development of semi-conductor technology has imparted practical interest in them at this time. Generally speaking, the transition from sine-cosine functions to other complete systems means a transition from linear, time-invariant components and equipment to linear, time-variable components and equipment, which, of course, constitute a much larger class. The mathematical theory of Walsh-Fourier analysis corresponds to the Fourier analysis used for sine-cosine functions. There is no theory of similar scope for block pulses, because they are incomplete.
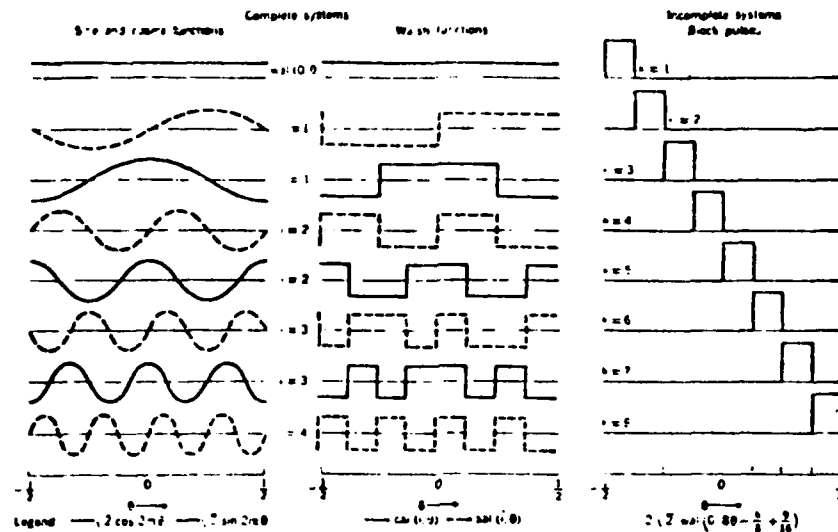
The sal and cal transforms of Walsh-Fourier analysis are defined by

$$a_s (\mu) = \int_{-\infty}^{\infty} F(\theta) \text{ sal } (\mu, \theta) \, d\theta$$

$$a_c (\mu) = \int_{-\infty}^{\infty} F(\theta) \text{ cal } (\mu, \theta) \, d\theta$$

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a)   Orthonormal systems of functions.



b)   List of features and applications of sine-cosine functions, Walsh
     functions, and block

| | Sine Functions | Walsh Functions | Block Pulses |
|---|---|---|---|
| Parameters | Amplitude<br>Frequency<br>Phase angle<br>... | Amplitude<br>Sequency<br>Delay<br>Time base | Amplitude<br>...<br>Pulse position<br>Pulse width |
| Mathematical theory | Fourier analysis | Walsh-Fourier analysis | ... |
| Power spectrum | Frequency spectrum | Sequency spectrum | ... |
| Filters | Time-invariable, linear | Periodically time variable, linear | Time variable, linear |
| Characterization | Frequency response of attenuation and phase shift | Sequency response of attenuation and delay | Attenuation as function of time |
| Multiples | Frequency division | Sequency division | Time division |
| Modulation | Amplitude, phase, frequency modulation | Amplitude, time position, time base, code modulation | Amplitude, pulse position, pulse width modulation |
| Radiable | sin 2πft, cos 2πft | $sal\left(i, \frac{t}{T}\right), cal\left(i, \frac{t}{T}\right)$ | ... |

$$F(\Theta) = \int_{-\infty}^{\infty} \left[ \alpha_s(\mu) \; sal(\mu \Theta) + \alpha_c(\mu) \; cal(\mu \Theta) \right] d\mu$$

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<u>Walsh-Function Filters</u> - For a sequency low pass filter based on Walsh functions the input signal, $F(\Theta)$, is transformed into a step function. $F\uparrow\uparrow(\Theta)$, with steps of a certain width, by integrating $F(\Theta)$ during an interval equal to the step width. The amplitudes of the steps are chosen so that $F\uparrow\uparrow(\Theta)$ yields a least-mean-square approximation of $F(\Theta)$. In addition, $F\uparrow\uparrow(\Theta)$ is delayed with respect to $F(\Theta)$ by one step width.

The number of samples obtained is equal to twice the cut-off sequency. Hence, the sampling theorems of Fourier analysis permit the comparison of frequency and sequency filters.

Theorems for the multiplications of Walsh functions have been proven. These are:

$$\text{cal } (k, \Theta) \text{ cal } (i, \Theta) = \text{cal} \left| k \oplus i, \Theta \right|$$
$$\text{sal } (k, \Theta) \text{ cal } (i, \Theta) = \text{sal} \left| i \oplus (k-1) + 1, \Theta \right|$$
$$\text{sal } (k, \Theta) \text{ sal } (i, \Theta) = \text{cal} \left| (k-1) \oplus (i-1), \Theta \right|$$

where the symbol $\oplus$ indicates modulo 2 addition. Note that the product of two Walsh functions yields only one Walsh function. Therefore, the amplitude modulation of a Walsh carrier yields only one sequency sideband as compared to the two sidebands obtained when a sine carrier is modulated. A typical application of the multiplication theorems of Walsh functions is in the design of sequency-bandpass filters.

<u>Digital Filtering and Multiplexing</u> - One of the most promising aspects of Walsh functions is the case with which filters and multiplex equipment can be implemented as digital circuits. The reason is that numerical Walsh-Fourier transformation and numerical sequency shifting of signals require summations and subtractions only. In the case of sine-cosine functions, the corresponding operations require multiplications with irrational numbers.

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A digital filter based on Walsh functions can be readily obtained. The input signal passes first through a sequency low-pass filter then transforms it into a step function. This step function is sampled and the samples are transformed into numbers by an analog/digital converter. A series of these numbers is stored in a digital storage. A Walsh-Fourier transform of this series is obtained by performing certain additions and subtractions in an arithmetic unit. Some or all of the obtained co-efficients, that represent sequency components, may be suppressed or altered - in effect, a filtering process. An inverse Walsh-Fourier trans-form yields the filtered signal as a series signal by digital/analog con-verter. Since there is a fast Walsh-Fourier transform just as ther is a fas Fourier transform, the arithmetic operations in a digital sequency filter are not only simpler than in a digital frequency but can be performed faster.

One of the features of Walsh functions that makes them of some interest in signal processing is the fact that their amplitudes are given precisely by a single bit, so that their use does not directly contribute to roundoff noise. The basis vectors of symmetry analysis offer the same attraction with, additionally, for low orders of input data frames N, some economy of computations by reason of the zeros.

## OTHER IMAGE TRANSFORMS

The Fourier transform is the transform most often used in image processing applications; there are other transforms which are also of interest in this area.

The one-dimensional, discrete Fourier transform is one of a class of important transforms which can be expressed in terms of the general relation

$$T(u) = \sum_{x=0}^{N-1} f(x)g(x,u) \tag{5-1}$$

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where $T(u)$ is the transform of $f(x)$, $g(x,u)$ is the forward transformation kernel, and u assumes values in the range 0, 1, ..., N-1. Similarly, the inverse transform is given by the relation

$$f(x) = \sum_{u=0}^{N-1} T(u)h(x,u) \qquad (5-2)$$

where $h(x,u)$ is the inverse transformation kernel and x assumes values in the ranges 0, 1, ..., N-1. The nature of a transform is determined by the properties of its transformation kernel.

For two dimensional square arrays the forward and inverse transforms are given by the equations

$$T(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)g(x,y,u,v) \qquad (5-3)$$

and

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} T(u,v,)h(x,y,u,v) \qquad (5-4)$$

where, as above, $g(x,y,u,v)$ and $h(x,y,u,v)$ are called the forward and inverse transformation kernels, respectively.

The two dimensional Fourier transform has the kernel

$$g(x,y,u,v) = \frac{1}{N} \exp\left[-j2\pi(ux + vy)/N\right]$$

which is separable and symmetric since

$$g(x,y,u,v) = g_1(x,u)g_1(y,v)$$

$$= \frac{1}{\sqrt{N}} \exp\left[-j2\pi ux/N\right] \frac{1}{\sqrt{N}} \exp\left[-j2\pi vy/N\right]$$

It is easily shown that the inverse Fourier kernel is also separable and symmetric.

A transform with a separable kernel can be computed in two steps, each requiring a one dimensional transform. First, the one dimensional transform is taken along each row of $f(x,y)$, yielding

$$T(x,v) = \sum_{y=0}^{N-1} f(x,y)g_2(y,v) \tag{5-5}$$

for $x,v = 0, 1, 2, \ldots, N-1$. Next, the one dimensional transform is taken along each column of $T(x,v)$; this results in the expression

$$T(u,v) = \sum_{x=0}^{N-1} T(x,v)g_1(x,u) \tag{5-6}$$

for $u,v = 0, 1, 2, \ldots, N-1$. The same final results are obtained if the transform is taken first along each column of $f(x,y)$ to obtain $T(y,u)$ and then along each row of the latter function to obtain $T(u,v)$. Similar comments hold for the inverse transform if $h(x,y,u,v)$ is separable.

If the kernel $g(x,y,u,v)$ is separable and symmetric, Eq. (5-3) can also be expressed in the following matrix form:

$$T = AFA \tag{5-7}$$

where F is the NXN image matrix, A is an NXN symmetric transformation matrix with elements $a_{ij} = g_1(i,j)$, and T is the resulting NXN transform for values of u and v in the range, $0, 1, 2, \ldots, N-1$.

To obtain the inverse transform we pre-multiply and post-multiply Eq. (5-7) by an inverse transformation matrix B.

If $B = A^{-1}$, it then follows that

$$F = BTB$$

which indicates that the digital image F can be recovered completely from its transform. If B is not equal to $A^{-1}$, then we obtain an approximation to F, given by the relation

$$F = BAFAB$$

A number of transforms, including the Fourier, Walsh, and Haar transforms, can be expressed in this form. An important property of the resulting transformation matrices is that they can be decomposed into products of matrices with fewer non-zero entries than the original matrix. This result, first formulated by Good (1958) for the Fourier transform, reduces redundancy and, consequently, the number of operations required to implement a two-dimensional transform. The degree of reduction is equivalent to that achieved by an FFT algorithm, being on the order of $N \log_2 N$ multiply/add operations for each row or column of an NXN image.

## Walsh Transform

When $N = 2^n$, the discrete Walsh transform of a function $f(x)$, denoted by $W(u)$, is obtained by substituting the kernel

$$g(x,u) = \frac{1}{N} \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-1-i}(u)} \tag{5-8}$$

into Eq. (5-1). In other words,

$$W(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \prod_{i=0}^{N-1} (-1)^{b_i(x)b_{n-1-i}(u)} \tag{5-9}$$

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where $b_k(z)$ is the kth bit in the binary representation of $z$. For example, if $n = 3$ and $z = 6$ (110 in binary), we have that $b_0(z) = 0, b_1(z) = 1$, and $b_2(z) = 1$.

The values of $g(x,u)$, excluding the 1/N constant term, are listed below for N = 8. The array formed by the Walsh transformation kernel is

Values of the Walsh trans-
formation kernel for N = 8.

| u\x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| 0 | + | + | + | + | + | + | + | + |
| 1 | + | + | + | + | − | − | − | − |
| 2 | + | + | − | − | + | + | − | − |
| 3 | + | + | − | − | − | − | + | + |
| 4 | + | − | + | − | + | − | + | − |
| 5 | + | − | + | − | − | + | − | + |
| 6 | + | − | − | + | + | − | − | + |
| 7 | + | − | − | + | − | + | + | − |

a symmetric matrix whose rows and columns are orthogonal. These properties, which hold in general, lead to an inverse kernel which is identical to the forward kernel, except for a constant multiplicative factor of 1/N. Thus, the inverse Walsh transform is given by

$$f(x) = \sum_{u=0}^{N-1} W(u) \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-1-i}(u)} \qquad (5\text{-}10)$$

Notice that, unlike the Fourier transform which is based on trigonometric terms, the Walsh transform consists of a series expansion of basis functions whose values are either plus or minus one.

It is also of interest to note that the forward and inverse Walsh transforms differ only by the 1/N term. Thus, any algorithm for computing the forward transform can be used directly to obtain the inverse transform simply by multiplying the result of the algorithm by N.

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The forward and inverse Walsh transforms are also equal given by

$$W(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-1-i}(u) + b_i(y)b_{n-1-i}(v)}$$

$$(5-11)$$

and

$$f(x,y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} W(u,v) \prod_{i=0}^{n-1} (-1)^{v_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)}$$

$$(5-12)$$

Thus, any algorithm which is used to compute the two dimensional forward Walsh transform can also be used without modification to compute the inverse transform.

The Walsh transform can be computed by a fast algorithm identical in form to the successive doubling method for the FFT. The only difference is that all exponential terms $W_N$ are set equal to one in the case of the fast Walsh transform (FWT).

The Walsh transform is real, thus requiring less computer storage for a given problem than the Fourier transform, which is in general complex valued.

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